

BORN-INFELD COSMOLOGIES

Ricardo García-Salcedo and Nora Bretón

*Depto. de Física, Cinvestav
A.P. 14-740, 07000 México D.F., México*

Received

We present a model for an inhomogeneous and anisotropic early universe filled with a nonlinear electromagnetic field of Born-Infeld (BI) type. The effects of the BI field are compared with the linear case (Maxwell). Since the curvature invariants are well behaved then we conjecture that our model does not present an initial big bang singularity. The existence of the BI field modifies the curvature invariants at $t = 0$ as well as sets bounds on the amplitude of the conformal metric function.

1. Introduction

The advantage in exploring a cosmological model with nonlinear electrodynamics resides in approaching with a classical model but taking into account some quantal features. It is well known that nonlinear electrodynamics (NLE) is the best classical solution to the self-energy problem of charged particles. The aim of nonlinear electrodynamics of Born and Infeld (BI) ¹ was to establish a model of a finite classical field theory without divergences. However, NLE can also be a model of quantum electrodynamics in the classical limit of high occupation numbers. As demonstrated by Stehle and De Baryshe ², theories with Lagrangians similar to the Heisenberg-Euler effective Lagrangians (the weak field expansion of the Born-Infeld theory is of this form) are more accurate classical approximations of QED than Maxwell's theory in the case of fields with high intensities at a fixed frequency. This fact makes NLE particularly interesting in general relativity, specially in cosmological theories, as a simple classical model to explain vacuum polarization processes- a possible influence on the mechanism of the evolution of the early universe. The processes called scattering of light by light can be expressed in terms of a kind of electric and magnetic permeability of vacuum. From this point of view, the exact solutions of the Einstein-Born-Infeld (EBI) equations are worth to study since they may indicate the physical relevance of nonlinear effects in strong gravitational and strong electromagnetic fields.

The Born-Infeld theory is self-consistent and satisfies all natural requirements. The BI Lagrangian depends on a constant b , that is a maximum field strenght, and also depends on the invariants of the field. Recently BI has recover interest in the

context of string theory, since the low energy action Lagrangian corresponding to the objects called *branes* can be written as a term that is BI plus a Weiss-Zumino or Chern-Simmons term ³. Furthermore, the BI strings are solitons which represent strings ending on branes.

In relation to exact solutions of the EBI equations we can mention that there are exact solutions for spaces possessing one timelike and one spacelike Killing vector fields. They correspond to spherical static spacetimes ⁴ and also to axisymmetric stationary spacetimes ⁵.

In the literature cosmological models have been proposed for the source-free Einstein-Maxwell equations ^{6, 7}. Charach and Malin ⁸ presented a cosmological model of Einstein-Rosen with gravitational, electromagnetic and scalar waves, this work generalized the electromagnetic Gowdy model. In all of these models the matter fields cause an evolution significantly different from that of the vacuum model. The interest is to get insight on the coupled electromagnetic and gravitational radiation at early epochs of the universe. Since at that epoch the fields must had been very strong, the nonlinear electromagnetic effects could be of importance. These fields could originate from processes near the Planck era ($t \sim 10^{-43}s$).

The spacetime we address in this work possesses two Killing vectors, both space-like. It is not immediate to conclude that this spacetime is also compatible with a BI source. For instance it does not admit a perfect fluid source together with BI, which is the case in the stationary axisymmetric spacetime. This is the first attempt, as far as we know, of exploring BI early universes. Our aim is to determine to what extent the BI field can prevent or modify the initial big bang singularity. Spatially inhomogeneous cosmological models with perfect fluid which do not originate in an initial singularity have been previously found by Senovilla ⁹ (see also ^{10, 11}). For the solutions derived in the present work the initial big bang singularity appears only as a coordinate singularity, since the invariants do not diverge at the origin. It turns out that the existence of the BI field set bounds to the amplitude of the conformal metric function and also affects quantitatively the curvature invariants at $t = 0$. In this work we do not treat the possible transition of our model to a homogeneous, isotropic one. In Sec. 2 the Einstein-Born-Infeld equations are established. In Sec. 3 the solution is presented and analyzed in the linear limit as well as the asymptotic behavior. In Sec. 3.4 we remark the coordinate transformations relating the spacetime studied and the Einstein-Rosen metrics. Concluding remarks are given at the end.

2. Einstein-Born-Infeld Equations

The line element considered corresponds to a G_2 spacetime, given by

$$ds^2 = \frac{1}{\varphi^2} \left\{ \frac{dz^2}{h} - \frac{dt^2}{s} + G[e^{-W} dy^2 + e^W (dx + mdy)^2] \right\}, \quad (1)$$

where x, y are ignorable coordinates (i.e. the spacetime has two spacelike Killing vectors, ∂_x, ∂_y), the metric functions depend on z and t : $\varphi = \varphi(t)$, $h = h(z)$,

$s = s(t)$. $G(z, t)$ and $e^W(z, t)$ are separable functions of z and t . The metric (1) corresponds to a space with gravitational waves propagating in the z direction, with their wavefronts on the plane surfaces spanned by the Killing vectors. The function W describes what is called the “+” polarization of gravitational waves. $G(z, t)$ describes the transverse scale expansion created by the energy density of the waves. The function $m(z) = m_0 + 2lz$ is playing the role of unpolarizer for the propagating gravitational waves that constitute the background. Making $l \rightarrow 0$ the metric recover the form of a space of linearly polarized gravitational waves (diagonal metric).

To interpret line element (1) as cosmological, we refer to the Einstein-Rosen metric ¹², Eq. (32), Sec. (3.3). The local behavior of these spacetimes is defined by the gradient of the metric function G , i.e. $G_{,\mu}$ which can be spacelike, null or timelike. The globally spacelike case corresponds to cylindrical spacetimes. The case of $G_{,\mu}$ being globally null corresponds to the plane symmetric waves. The other case of $G_{,\mu}$ being globally timelike as well as cases in which the sign of the gradient can vary from point to point are used to describe cosmological models and also colliding gravitational waves ¹³. It must also be remarked that due to the presence of an Abelian subgroup G_2 in the Bianchi models of types I-VII, the Einstein-Rosen metrics include these models as particular cases ¹⁴. Also the rotationally invariant Bianchi models of types VIII and IX belong to the generalized Einstein-Rosen spacetimes ¹⁵.

We note that in the case of Einstein-Rosen spacetimes, the interpretation as gravitational waves propagating in the z direction arises directly from the equation satisfied by G ,

$$G_{,tt} - G_{,zz} = 8\pi G e^f (T_a^a - T), \quad (2)$$

For vacuum or when $T_a^a = T$, G satisfies the usual wave equation. In our case, the B-I field is not traceless and then G is some propagating wave with a source (the B-I field).

Turning back to the spacetime (1), it is filled with a BI field with energy-momentum tensor given by:

$$T_{\mu\nu} = 2\mathcal{H}_{,P} P_{\mu\alpha} P^\alpha{}_\nu - g_{\mu\nu} [2P\mathcal{H}_{,P} + Q\mathcal{H}_{,Q} - \mathcal{H}], \quad (3)$$

where \mathcal{H} is the Born-Infeld structural function and $\mathcal{H}_{,x} := \partial\mathcal{H}/\partial x$. P and Q are the invariants associated with the Born-Infeld field, given by

$$P = \frac{1}{4} P_{\mu\nu} P^{\mu\nu}, \quad Q = \frac{1}{4} P_{\mu\nu} \check{P}^{\mu\nu}, \quad (4)$$

$\check{P}^{\mu\nu}$ denotes the dual tensor of $P^{\mu\nu}$, defined by $\check{P}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} P_{\alpha\beta}$. The antisymmetric tensor $P_{\mu\nu}$ is the generalization of the electromagnetic tensor $F_{\mu\nu}$. The structural function is constrained to satisfy some physical requirements: (i) the correspondence to the linear theory [$\mathcal{H}(P, Q) = P + O(P^2, Q^2)$]; (ii) the parity conservation [$\mathcal{H}(P, Q) = \mathcal{H}(P, -Q)$]; (iii) the positive definiteness of the energy density

($\mathcal{H}_{,P} > 0$) and the requirement of the timelike nature of the energy flux vector ($P\mathcal{H}_{,P} + Q\mathcal{H}_{,Q} - \mathcal{H} \geq 0$). Such structural function for the BI field is given by

$$\mathcal{H} = b^2 - \sqrt{b^4 - 2b^2P + Q^2}, \quad (5)$$

where b is a maximum field strenght of the BI field. In the linear limit, which is obtained by taking $b \rightarrow \infty$, then $\mathcal{H} = P$ and $P_{\mu\nu} = F_{\mu\nu}$, coinciding with the Maxwell electromagnetism. $P_{\mu\nu}$ and $F_{\mu\nu}$ are related through the material or constitutive equations:

$$F_{\mu\nu} = \mathcal{H}_{,P}P_{\mu\nu} + \mathcal{H}_{,Q}\tilde{P}_{\mu\nu}, \quad (6)$$

We have worked out the problem in the null tetrad formalism¹⁶. The function \mathcal{H} in Eq. (5) involves a square root that complicates the solving of the system of the Einstein-Born-Infeld equations. To surmount this difficulty, first of all we have aligned the two non-null components of $P_{\mu\nu}$ along the two principal directions of the tetrad vectors e^3 and e^4 (the metric is type D); second, we parametrize the BI field in terms of the electric and magnetic fields: $P_{34} = D$ and $P_{12} = iH$ where D and H stand for the usual electric displacement and magnetic field intensity, respectively. Also, we introduce the function $\nu = \nu(D, H)$,

$$\exp[2\nu] = \frac{b^2 + D^2}{b^2 - H^2}, \quad (7)$$

If the previous Eq. (7) has to make sense, it demands that $b > H$. From the structural Eqs. (6) we obtain the expressions that clearly manifest the vacuum polarization, in this case depending on the function $\nu(z, t)$:

$$D = e^\nu E, \quad B = e^\nu H, \quad (8)$$

Besides the Einstein field equations we must consider the BI electrodynamical equations that amount to the closure condition of the electromagnetic two form, ω ,

$$d\omega = d\{(F_{12} + P_{34})e^1 \wedge e^2 + (F_{34} + P_{12})e^3 \wedge e^4\} = 0, \quad (9)$$

This formalism of the Born-Infeld field was introduced by Plebański et al¹⁷. We have assumed that $h(z) = \alpha + \beta z + \epsilon z^2$, where ϵ can take the values $-1, 0, 1$. For each value of ϵ we obtain different curvatures in the z direction. For the metric (1) and with the alignment chosen the electromagnetic equations are

$$(E + iH)_{,z} = 0, \quad (10)$$

$$(\ln(D + iB))_{,t} = -2ile^{-\nu} + 2(\ln \varphi)_{,t}, \quad (11)$$

Eq.(9) tells us that neither E nor H depend on z . Eq.(10) and Eq. (15), quoted below, lead to infer that neither D nor B depend on z . Therefore, for this spacetime, the BI is a spatially homogeneous field. The Einstein equations for the BI energy-momentum tensor (3) amount to

$$\varphi_{,tt} + l^2\varphi = 0, \quad (12)$$

$$\frac{\varphi^2}{2}s_{,tt} - 2\varphi\varphi_{,t}s_{,t} + s(3\varphi_{,t}^2 + 3l^2\varphi^2) = 2b^2(e^{-\nu} - 1), \quad (13)$$

$$\varphi\varphi_{,t}s_{,t} + s(-3\varphi_{,t}^2 + l^2\varphi^2) + \epsilon\varphi^2 = -2b^2(e^\nu - 1) \quad (14)$$

It is a system of equations for the functions $\varphi(t)$, $\nu(t)$ and $s(t)$, while the rest of the metric functions are $e^W = (s/h)^{\frac{1}{2}}$ and $G(z, t) = (hs)^{\frac{1}{2}}$. Eq. (11) for $\varphi(t)$ can be solved immediately, giving $\varphi(t) = A \cos(lt) + B \sin(lt)$, with A, B constants. Eqs. (12)-(13) can be decoupled and it is obtained the solution for $\nu(t)$ in terms of $\varphi(t)$ as well as Eq. (16) for $s(t)$:

$$\nu(t) = \frac{1}{2} \ln(1 - \varphi(t)^4), \quad (15)$$

$$\varphi \varphi_{,t} s_{,t} + s(-3\varphi_{,t}^2 + l^2 \varphi^2) + \epsilon \varphi^2 = -2b^2(\sqrt{1 - \varphi^4} - 1), \quad (16)$$

In order that $\nu(t)$, given in Eq. (15), be a real function, $\varphi(t) = A \cos(lt) + B \sin(lt)$ must be such that $|\varphi|^4 < 1$. This means that the existence of the BI field restrains the conformal metric function $\varphi(t)$ in its amplitude because it bounds the values of the constants A and B . Demanding that

$$\begin{aligned} A < 1 & \quad \text{if } B = 0; \\ B < 1 & \quad \text{if } A = 0; \\ A^2 + B^2 < 1 & \quad \text{if } A \neq 0, \quad B \neq 0, \end{aligned} \quad (17)$$

fulfils the requirement $|\varphi|^4 < 1$. Putting the NLE functions, $D(t)$ and $B(t)$, Eqs. (8), in terms of $\varphi(t)$

$$\begin{aligned} D(t) &= \varphi^2 \cos[2l \int (1 - \varphi^4)^{-\frac{1}{2}} dt], \\ B(t) &= -\varphi^2 \sin[2l \int (1 - \varphi^4)^{-\frac{1}{2}} dt], \end{aligned} \quad (18)$$

with the corresponding coordinate components of the electromagnetic field $F_{\mu\nu}$,

$$\begin{aligned} F_{tx}(t) &= e^{-\nu} \cos[2l \int (1 - \varphi^4)^{-\frac{1}{2}} dt], \\ F_{zy}(t) &= -e^{\nu} \sin[2l \int (1 - \varphi^4)^{-\frac{1}{2}} dt], \end{aligned} \quad (19)$$

The previous expressions show that the BI field modulates the amplitude and frequency of the electromagnetic field. The fields $B(t)$ and $D(t)$ are shown in Fig. 1. They are oscillating functions depending on the constant b through the function e^{ν} .

Since the metric is of Petrov type D, the only nonvanishing Weyl scalar is given by

$$\psi_2 = \varphi^2 \left[\frac{s_{,tt}}{4} - \frac{\epsilon}{2} - 2l^2 s + i \frac{3l}{2} s_{,t} \right], \quad (20)$$

This scalar is not real and therefore the magnetic part of the Weyl tensor does not vanish. We also note that it is homogeneous, i. e. it depends only on time. The invariants are given by $I = 3\psi_2^2$, $J = -\psi_2^3$; from their expressions it is clear

that they share (if they exist) the singularities of ψ_2 . On its turn ψ_2 depends on the behaviour of the functions $\varphi(t)$ and $s(t)$. $\varphi(t)$ is a periodic function while $s(t)$, governed by Eq. (16), carries much of NLEBI information. In the next section we analyze the behavior of $s(t)$.

3. The BI solution and the linear case

The solution to Eq. (16) for $s(t)$, in terms of φ is given by

$$s(t) = \varphi^3 \varphi_{,t} \{c_1 - \int \frac{2b^2(\sqrt{1-\varphi^4} - 1) + \epsilon \varphi^2}{\varphi^4 \varphi_{,t}^2} dt\}, \quad (21)$$

where c_1 is an integration constant related to initial or boundary conditions for vacuum. It is interesting now to analyze graphics of the function $s(t)$. Fig. 2 shows $s(t)$ for several values of the BI constant b . For values of $b > 1$ the function $s(t)$ becomes positive for all the range and reduces its oscillations; for larger b the maximums become greater. The transverse scale expansion $G(z, t) = \sqrt{h(z)s(t)}$ is an inhomogeneous function expanding in the z direction, it is shown in Fig.3.

Fig. 4 shows the plots of the real and imaginary parts of ψ_2 ; they give us qualitative information with respect to the singularities of the invariants I and J . Both $\text{Re}\psi_2$ and $\text{Im}\psi_2$ are continuous functions that do not present infinities at all.

In Fig. 5, the function $s(t)$ displays different behaviours at $t = 0$ depending if $\varphi = B \sin(lt)$ or $\varphi = A \cos(lt)$. In the former case $s(0) = 0$ and consequently there appears a coordinate singularity at $t = 0$; however the corresponding Weyl scalar is not singular there. We can see from Fig. 5 that this class of solutions admits both behaviors at $t = 0$, with coordinate singularity or without it. To a linear combination $\varphi = A \cos(lt) + B \sin(lt)$ corresponds $s(0) \neq 0$.

3.1. Linear limit

In order to compare the effects of the BI field with the Maxwell case, we compute the linear limit. The Maxwell electrodynamics is recovered when $b \rightarrow \infty$. The BI Eqs. (9) and (11) in the linear limit correspond to

$$[\ln(D + iB)]_{,t} - 2(\ln \varphi)_{,t} = 0, \quad (22)$$

whose solution is

$$D = E = C \cos(2Alt), \quad B = H = C \sin(2Alt), \quad (23)$$

where A and C are constants. In terms of the coordinate components of the electromagnetic field the expressions are

$$F_{tx} = C \cos(2lt), \quad F_{zy} = -C \sin(2lt), \quad (24)$$

This electromagnetic field is homogeneous of intensity $C^2 = E^2 + H^2$. In accordance with the parametrization (7), for $b \rightarrow \infty$, Eqs. (12)-(13) become

$$\frac{\varphi^2}{2} s_{,tt} - 2\varphi \varphi_{,t} s_{,t} + s(3\varphi_{,t}^2 + 3l^2 \varphi^2) = -C^2,$$

$$\varphi\varphi_{,t}s_{,t} + s(-3\varphi_{,t}^2 + l^2\varphi^2) + \epsilon\varphi^2 = -C^2, \quad (25)$$

Solving for $s(t)$ in terms of $\varphi(t)$ we have

$$s(t) = \varphi^3\varphi_{,t}[c_1 - \int \frac{C^2 + \epsilon\varphi^2}{\varphi_{,t}^2\varphi^4}dt], \quad (26)$$

expression that also can be obtained from Eq.(21) taking the limit $b \rightarrow \infty$,

$$\lim_{b \rightarrow \infty} 2b^2(\sqrt{1 - \varphi^4} - 1) = \lim_{b \rightarrow \infty} 2b^2(e^\nu - 1) = C^2.$$

Fig. 6 shows the comparison between $s(t)$ corresponding to the linear electromagnetic field, Eq. (26), and $s(t)$ of the BI field, Eq. (21). The presence of BI field smooths the function $s(t)$ respect of the linear version. Making $C = 0$ we get the vacuum limit of the spacetime (1), by this meaning that only gravitational waves remain.

Comparing Eq.(21) for $s(t)$ corresponding to the BI field with Eq.(26) for the linear case we see that the first term in the integral is directly associated with the NLEBI effects. To show the dependence on b we write $s(t)$ as

$$s(t) = \varphi^3\varphi_{,t}[c_1 - \int \frac{f(b) + \epsilon\varphi^2}{\varphi_{,t}^2\varphi^4}dt], \quad (27)$$

where $f(b) = 2b^2(e^\nu - 1) = 2b^2\{\sqrt{\frac{b^2 + D^2}{b^2 - H^2}} - 1\}$, in accordance with the requirement in Eq. (7), the range of b is $b > H$. If one plots $f(b)$ vs. b , it can be seen that the NLEBI effects are sensible for values of b near H , while for larger b 's, generically it tends to the constant linear Maxwell field $C^2 = E^2 + H^2$.

3.2. Behavior at early times

For $t \ll 1$ it is reasonable to approximate $\varphi(t)$ as

$$\varphi(t) = A \cos(lt) + B \sin(lt) \approx A + Blt, \quad (28)$$

Since $\varphi = \text{const.}$ implies $s(t) = 0$, we take $\varphi \approx t$ along with the restrictions (17) imposed by the BI field. The constant l is supposed not to be large since we are in a low frequency regime. To solve Eq. (21) we also approximate $\sqrt{1 - \varphi^4} \approx 1 - \frac{\varphi^4}{2}$, obtaining

$$s(t) \approx \epsilon t^2 + c_1 t^3 + b^2[t^4 + O(t^5)], \quad (29)$$

predominance of one term over the others depend on the relative values between c_1 and b . The term in brackets represents the BI contribution at early times. It is expected b to be large if the electromagnetic field is strong, then the BI contribution can be of importance. The first term corresponds to the contribution of the space curvature, determined by the value of ϵ , $(1, -1, 0)$. The Weyl scalar ψ_2 approaches $t = 0$ as

$$\psi_2 = \frac{3c_1}{2}t^3 + (3b^2 - 2l^2\epsilon)t^4 + O(t^5) + i(3\epsilon lt^3 + \frac{9c_1 l}{2}t^4 + O(t^5)). \quad (30)$$

The contribution of the BI field goes as t^4 and higher orders in t . In the magnetic part of ψ_2 the contribution is not important at very early times since comes from the term with $\epsilon = -1, 0, 1$ and goes as t^3 .

We now mention the case $l = 0$, which corresponds to a spacetime of propagating linearly polarized gravitational plane waves (diagonal metric). Eq. (11) for $\varphi(t)$ become $\varphi_{,tt} = 0$, with solution $A_0 t + B_0$, A_0, B_0 being constants. The refore, the behavior of φ in the case $l = 0$ resembles the one for $t \ll 1$. Consequently, we assert that this spacetime approaches the origin in time as propagating linearly polarized waves.

3.3. Behavior at $t \gg 1$

To complete the previous analysis, the question arises as how does the model evolve at $t \gg 1$, or which is the asymptotic behavior of the solution at $t \gg 1$, i. e. for times when the universe is already a causally connected one.

The dynamics of our model is driven by the function $\varphi(t)$. For $t \gg 1$ both φ and $\varphi_{,t}$ become rapidly oscillating periodic functions that can be approximated by a constant. With this, the metric function $s(t)$ (Eq. (21) becomes a linear function on t . Absorbing constants and transforming $\sqrt{K_1 - K_2 t} \rightarrow K_2 T/2$ (K_1, K_2 being constants), the line element can be written as

$$ds^2 = \frac{dz^2}{h(z)} - dT^2 + h(z)dy^2 + \frac{K_2 T}{2}(dx + mdy)^2, \quad (31)$$

While the nonlinear electromagnetic field for $t \gg 1$, from Eq. (15), we see that $\nu(t)$ goes to a constant. Then, from Eqs. (18) for the fields $D(t), B(t)$, we recover the Maxwell case.

3.4. Relation to Einstein-Rosen Universes

By means of a coordinate transformation, the metric (1) can be put in the form of an Einstein-Rosen line element ¹²

$$ds^2 = \frac{1}{\varphi^2} \{ dZ^2 - dT^2 + G[e^{-W} dy^2 + e^W (dx + mdy)^2] \}. \quad (32)$$

The coordinate transformation for $z \rightarrow Z$ depends on the value of ϵ as follows

$$z = \left(\frac{\beta^2}{4} - \alpha \right)^{\frac{1}{2}} \cosh Z - \frac{\beta}{2}, \quad \epsilon = 1, \quad (33)$$

$$z = \left(\frac{\beta^2}{4} + \alpha \right)^{\frac{1}{2}} \sin Z + \frac{\beta}{2}, \quad \epsilon = -1, \quad (34)$$

$$z = \frac{Z^2 \beta}{4} - \frac{\alpha}{\beta}, \quad \epsilon = 0, \quad (35)$$

We also note that for $\epsilon = -1$, the functions in terms of the spatial coordinate z become periodic, then the topology of the universe in this case can be a closed one. The coordinate transformation for t , $\frac{dt^2}{s} \rightarrow dT^2$, is not a simple one in the general

case due to the form of $s(t)$, Eq. (21). It involves elliptic integrals which lead to Jacobi family of elliptic functions¹⁸ and there is no analytical expression in terms of elementary functions. However, for particular cases such transformation can be given in a simple form. An example is for early times. In this case the relation between t and T is $\exp(\sqrt{\epsilon}T) = t^{-2}(2\epsilon t + c_1 t^2 + 2t\sqrt{\epsilon(\epsilon + c_1 t + b^2 t^2)})$.

Another example is the case $b = 0$ (vacuum) and $\epsilon = 0$. In this case the line element (32) takes the form

$$ds^2 = \frac{1}{\varphi^2}(dZ^2 - dT^2) + TZ[e^{-W}dy^2 + e^W(dx + mdy)^2], \quad (36)$$

Gowdy¹⁹ constructed exact vacuum solutions of the Einstein field equations which represent inhomogeneous closed universes. These models possess compact spacelike hypersurfaces as well as G_2 invariance. The line element (36) can be written as a Gowdy model with three-torus topology performing the coordinate transformation

$$G = TZ \rightarrow \xi, \quad \frac{Z^2}{2} + \frac{T^2}{2} \rightarrow \zeta, \quad (37)$$

then one obtains the three-torus Gowdy line element

$$ds^2 = \frac{\varphi^{-2}}{2\sqrt{\zeta^2 - \xi^2}}(d\zeta^2 - d\xi^2) + \xi[e^{-W}dy^2 + e^W(dx + mdy)^2], \quad (38)$$

If one arrives to a Gowdy model for vacuum, one can speculate about that a geometry so simple as Gowdy does not admit a Born-Infeld field.

4. Concluding Remarks

In this work we have presented a family of solutions to the Einstein-Born-Infeld equations for a space-time which is a G_2 cosmological model. The spacetime describes propagating gravitational plane waves coupled with a nonlinear spatially homogeneous electromagnetic field of the Born-Infeld type. We also present the limit in which the BI field becomes Maxwell electromagnetism.

Some results are:

The presence of an initial coordinate singularity depends on the choice of the conformal metric function $\varphi(t) = A \cos(lt) + B \sin(lt)$, for $B \neq 0, A = 0$ the solution exhibits initial singularity while if $B = 0, A \neq 0$ there is no such coordinate singularity. However, from the smooth behavior of the Weyl scalar, ψ_2 , we guess that there are no singularities at all.

The presence of the BI field sets bounds on the amplitude of the periodic function $\varphi(t)$. In this sense BI modifies the global expansion of the spacetime. If compared with the effect of linear electrodynamics, the presence of the BI field smooths the metric function $s(t)$ (see Fig. 6). Since $s(t)$ is involved in the expression of the scale expansion and of the Weyl scalars, then BI field smooths the curvature. The BI field also modifies quantitatively the curvature at $t = 0$, the effect being more sensible when the BI parameter b approaches the magnitude of a critical magnetic

field intensity H . We conjecture that a less restrictive spacetime (not type D) shall permit the existence of an inhomogeneous nonlinear electromagnetic field, whose consequences on the space curvature could be more drastic.

For early times, $t \rightarrow 0$, the spacetime approaches a space of unpolarized gravitational waves. For $t \gg 1$ the spacetime becomes an inhomogeneous anisotropic spacetime with a homogeneous Maxwell field.

For vacuum one obtains a Gowdy model of three-torus topology.

It should be of interest to classify the spacetime in different regions according to the sign of the gradient of the function $G(z, t)$ and see the distinct interpretations of the found solution according to each region in the (z, t) plane.

Generalizing this solution to include a scalar field and to investigate the coupled effect with nonlinear electrodynamics for early universes could also lead to interesting results.

Acknowledgements

We gratefully acknowledge to the anonymous referee whose opinions help to improve this work. Partially supported by CONACyT (México), project No. 32086-E.

References

1. M. Born and L. Infeld, *Proc. Royal Soc. London* **A144**, 425 (1934).
2. P. Stehle and P. G. de Baryshe, *Phys. Rev.* **152**, 1135 (1966).
3. A. A. Tseytlin, *Nucl. Phys.* **B469**, 51 (1996).
4. R. Pellicer and R. J. Torrence, *J. Math. Phys.* **10**, 1718 (1969).
5. A. García, H. Salazar and J. F. Plebański, *Nuovo Cimento* **84**, 65 (1984).
6. M. A. Melvin, *Phys. Rev.* **139**, B225 (1965); *Phys. Lett.* **8**, 65 (1964). K. S. Thorne, *Phys. Rev.* **138**, B251 (1965); *Phys. Rev.* **139** B244 (1965).
7. M. Carmeli, Ch. Charach and S. Malin, *Phys. Rep.* 76, 79 (1981); Ch. Charach, *Phys. Rev.* **D 19**, 3516 (1979).
8. Ch. Charach and S. Malin, *Phys. Rev.* **D21**, 3284 (1980).
9. J. M. M. Senovilla, *Phys. Rev. Lett.* **64**, 2219 (1990).
10. M. Mars, *Phys. Rev.* **D51**, 3989 (1995).
11. O. Pimentel, *Mod. Phys. Lett. A* **14**, 43 (1999).
12. A. Einstein and N. Rosen, *J. Franklin Inst.* **223**, 43 (1937).
13. M. Carmeli and Ch. Charach, *Foundations of Phys.* **14**, 963 (1984).
14. K. Tomita, *Progr. Theor. Phys.* **59**, 1150 (1978).
15. M. Carmeli, Ch. Charach and A. Feinstein, *Ann. Phys. (N. Y.)*, **150**, 392 (1983).
16. J. Plebański, *Spinors, Tetrads and Forms* Ed. Cinvestav, I.P.N. (México, 1973).
17. H. Salazar, A. García and J. Plebański, *J. Math. Phys.* **28**, 2171 (1987).
18. N. Bretón, A. García, A. Macías and G. Yáñez, *J. Math. Phys.* **39**, 6051 (1998).
19. R. Gowdy, *Phys. Rev. Lett.* **27**, 827 (1971); *Ann. Phys. (N. Y.)* **83**, 203 (1974).

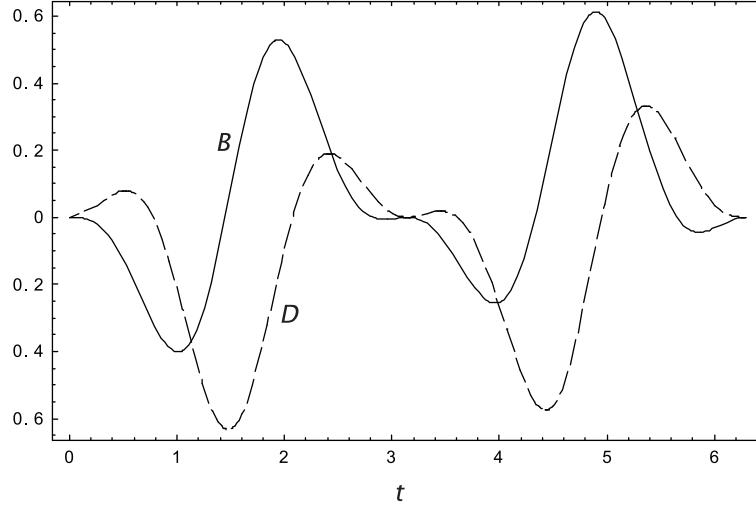


Fig.1. These are plots of the nonlinear electromagnetic fields, electric displacement, $D(t)$, and magnetic induction, $B(t)$. The corresponding conformal metric function is $\varphi = 0.8 \sin(t)$.

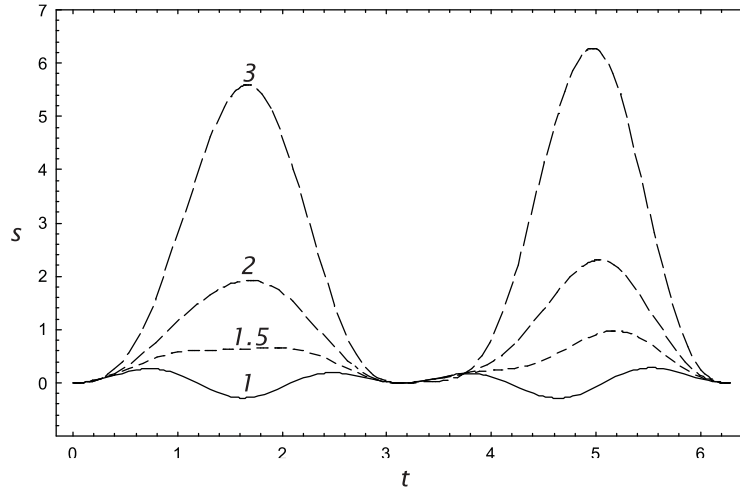


Fig.2. It is displayed the metric function $s(t)$ for different values of the BI constant b , $b = 1, 1.5, 2, 3$. For greater values of b the maximums are higher but the function preserves the shape of $b > 2$. These plots correspond to $\varphi = 0.8 \sin(t)$, $c_1 = 1$ and $\epsilon = 1$.

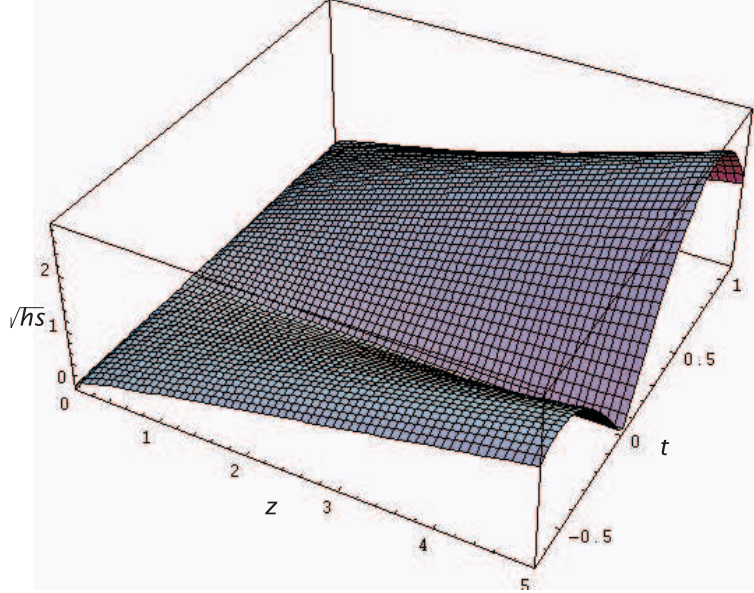


Fig.3. It is shown the transverse scale expansion $G(z, t) = \sqrt{h(z)s(t)}$. It is increasing in z and oscillating in t . For this plot the values of the constants are $\epsilon = 1, \beta = 1, \alpha = 0, c_1 = 1, b = 1$.

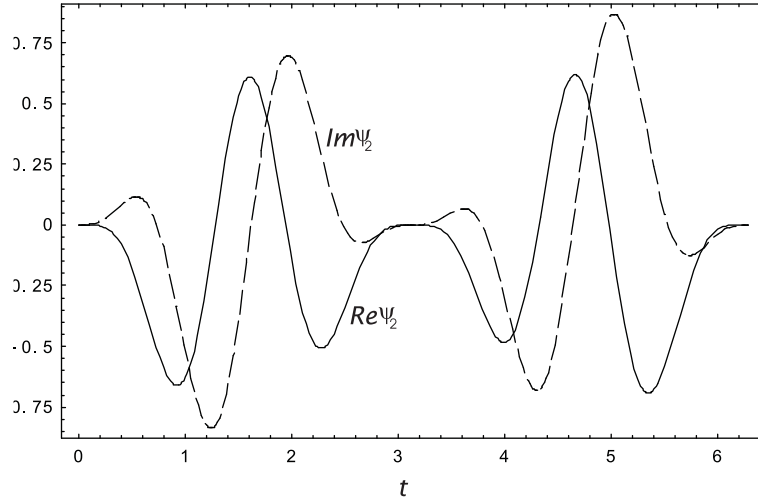


Fig.4. The continuous plot corresponds to $\text{Re}\psi_2$ and the dashed one to $\text{Im}\psi_2$. Both are continuous functions, on this basis we guess that the spacetime has no singularities. These graphics correspond to $s(t)$ in Fig.2 with $b = 1$.

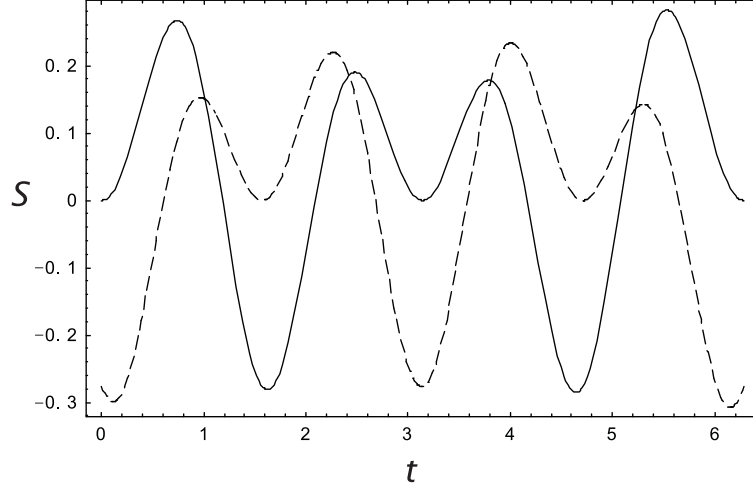


Fig.5. The function $s(t)$ displays different behaviour at $t = 0$ depending if $\varphi = 0.8 \sin(t)$ (continuous curve) or $\varphi = 0.8 \cos(t)$ (dashed curve).

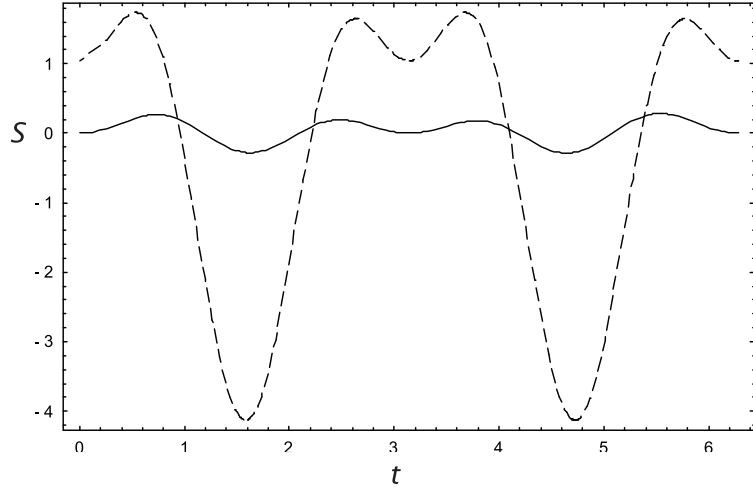


Fig.6. The function $s(t)$ for the linear limit corresponds to the dashed curve, the continuous line is $s(t)$ in the presence of BI field with $b = 1$. The constant $c_1 = 1$. $s(t)$ of BI field is smoother than the one corresponding to linear electromagnetic field. The conformal metric function considered is $\varphi = 0.8 \sin(t)$.